Kinematics

Yaouen Fily

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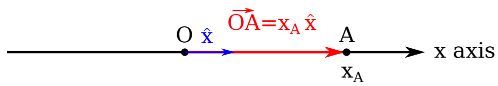
Contents

1 1D Motion

In this section we consider an object that is restricted to a line. It can move forward, backward, or stop, but only ever along that straight line. There is no need for a y axis here, a single axis (x, along the line on which the object moves) is enough.

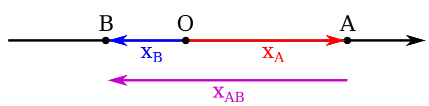
1.1 Position, displacement, distance

Since there's only one axis and one position coordinates, the position is really just a number. It's still useful to think of those numbers as vector arrows (to keep track of signs, and to help make the jump to 2D motion later), but we don't use the full vector notation, which would be unnecessarily cumbersome.



Rather than writing the position vector of A as $\overrightarrow{OA} = x_A\hat{x}$, we just write that the position of A is x_A.

The position of B relative to A, which we may also call the displacement from A to B if A and B are positions of the same object at two different times, is defined the same way we defined those quantities in 2D: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A)\hat{x}$, which we just write $x_{AB} = x_B - x_A$. As discussed before, the rationale behind this formula is that to go from A to B we need to travel from A to O, which is O to A in reverse (hence the minus sign in front of the coordinate of x_A), then from O to B. This formula works regardless of the signs of x_A, x_B, and x_{AB}. In the example below, $x_A = 2$ and $x_B = -1$. Then $x_{AB} = (-1) - (2) = -3$, which is negative because you have to travel left (against the axis' orientation, which in this case is to the right, indicated by the black arrow at the right end of the axis), and the distances "add up" because the points are on opposite sides of the origin. You don't need to worry about that though, the formula takes care of signs automatically. I'm just highlighting the fact that the result makes sense.



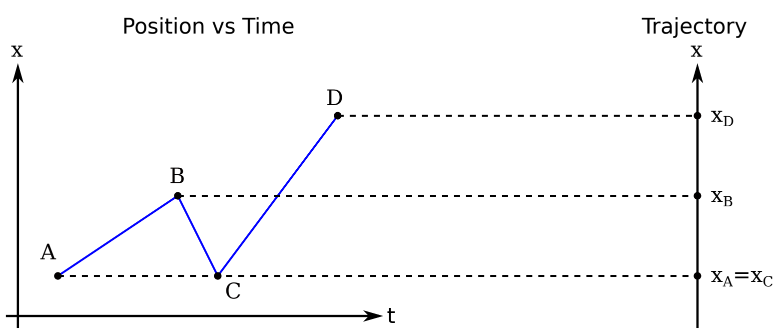
The distance between A and B is $|x_A B|$, which is always positive, unlike x_{AB}. Note that this is a natural way to extend the 2D definition of the magnitude: $\|\overrightarrow{AB}\| = \sqrt{x_{AB}^2 + 0} = |x_{AB}|$ where the zero represents the "missing" y component.

If you have more than two displacements, you can write something like $x_{AB} + x_{BC} + x_{CD} = x_{AD}$, which is the 1D version of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$. You can read this as: the net result of going from A to B, then from B to C, then from C to D, is that you've gone from A to D.

1.2 Position vs time

The position x of an object as it moves can be thought of as a function of time: $x(t)$.

Problem 1: Position vs time and trajectory.



1. The graph on the left shows the x coordinate of an object as a function of time. The vertical axis on the right show the x axis along which the object is physically moving. Sketch the trajectory of the object on the axis on the right, i.e., the sequence of places where it goes. (If the object passes through the same point twice, draw the second time a little off the axis so you can tell the two passes apart).
2. Compare the distance traveled from A to B and the distance traveled from B to C. Compare the time taken to travel from A to B a(nd time taken to travel from C to D. What can you conclude?

1.3 Velocity and speed

Average velocity

The average velocity, or velocity between two times t_1 and t_2, is defined as the displacement divided by the time elapsed:

$$\bar{v}(t_1,t_2)) = \frac{x(t_2)-x(t_1)}{t_2-t_1}.$$

Instantaneous velocity

The instantaneous velocity at time t_1 is the limit of the average velocity between t_1 and t_2 goes to t_1:

$$v(t_1) = \lim_{t_2\to t_1} \bar{v}(t_1,t_2) = \lim_{t_2\to t_1} \frac{x(t_2)-x(t_1)}{t_2-t_1}.$$

If we rename t_1 to t and define $\Delta t = t_2 - t_1$, we can also write this as

$$v(t) = \lim_{\Delta t\to 0} \bar{v}(t,t+\Delta t) = \lim_{\Delta t\to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}.$$

At this point you should be realizing that $v(t)$ is none other than the derivative of $x(t)$ with respect to time:

$$v(t) = \frac{dx}{dt}.$$

Note: We won't always specify average or instantaneous. Sometimes you'll have to guess from context. Similarly, the average velocity doesn't always have a bar above the v.

Graphical interpretation

Speed

Speed is a distance traveled per unit time. Like distance, but unlike velocity, it is always positive.

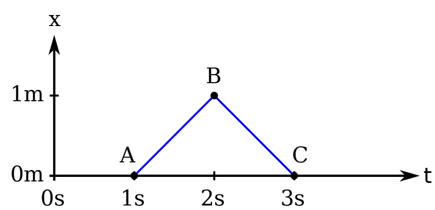
The average speed between two times t_1 and t_2 is the distance traveled during that time divided by the time elapsed $(t_2 - t_1)$.

The instantaneous speed at a time t is the limit of the average speed between t and $t + \Delta t$ when $\Delta t \to 0$. It is to the average speed what the instantaneous velocity is to the average velocity.

In 1D, the instantaneous speed is the absolute value of the instantaneous velocity:

$$\left|\frac{dx}{dt}\right| = \left|\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}\right| = \lim_{\Delta t \to 0} \frac{|x(t+\Delta t) - x(t)|}{\Delta t} \equiv \lim_{\Delta t \to 0} \frac{\text{distance}}{\text{time elapsed}} \equiv \text{speed}$$

However, the average speed is not always the absolute value of the average velocity. For example:



The displacement between $t = 1\,\text{s}$ and $t = 3\,\text{s}$ is $\Delta x = x(3\,\text{s}) - x(1\,\text{s}) = 0\,\text{m}$, but the distance traveled is $|x(2\,\text{s}) - x(1\,\text{s})| + |x(3\,\text{s}) - x(2\,\text{s})| = 2\,\text{m}$. To compute the distance we had to split the motion into stretches with constant direction, compute the distance traveled during each, then add them up. As a result, the average velocity between $t = 1\,\text{s}$ and $t = 3\,\text{s}$ is $0\,\text{m/s}$, but the average speed is $\frac{2\,\text{m}}{2\,\text{s}} = 1\,\text{m/s}$.

Problem 2: 1D velocity.

The position x of an object changes through time according to $x(t) = t - 2t^2 + t^3$.

1. Graph $x(t)$ between $t = 0$ and $t = 1.5$. If you don't have a graphing program or calculator with you, use https://www.desmos.com/calculator.
2. Compute the average velocity between $t = 0$ and $t = 0.5$.
3. Compute the instantaneous velocity at $t = 0.2$ and at $t = 0.5$.
4. Sketch the graph from question 1. Where do the three velocities you just computed show up? Add them to the sketch.
5. Compute the largest x reached between $t = 0$ and $t = 1$. What is the net displacement between $t = 0$ and $t = 1$? What is the distance traveled between $t = 0$ and $t = 1$?
6. What is the average speed between $t = 0$ and $t = 1$?

1.4 Uniform motion (constant velocity)

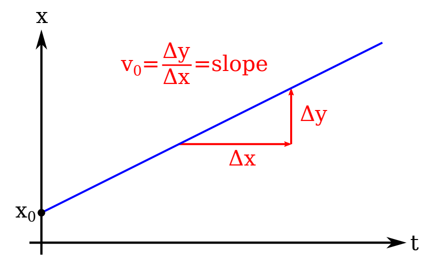
A uniform motion is one in which the velocity is constant.

The position-vs-time function in a uniform motion is

$$x(t) = x_0 + v_0 t$$

where x_0 is the position at $t = 0$ (you can check that setting $t = 0$ does indeed yield $x(t) = x_0$) and v_0 is the constant velocity.

Note: The subscript "0" is often used to convey that something is constant. The meaning of "constant" depends on context. In this case, it highlights that x_0 and v_0 do not change over the course of the motion.



Problem 3: Uniform 1D motion.

Use the definition of 1D uniform motion above to compute the average velocity between two times t_1 and t_2, then the instantaneous velocity at a time t. Show that they're all equal and independent of time.

Problem 4: Sketching the position and velocity.

For each of the following motions, sketch $x(t)$ and $v(t)$. Assume $x(0) = 0$. Note: There more than one correct way to draw each graphs; just make sure your graphs are consistent with the available information.

1. Moving forward at 1 m/s for 3 s, then backward at 2 m/s for 1 s.
2. Moving forward, continuously slowing to a stop, then remaining static for a bit, then gaining speed in the backward direction.
3. Moving backward, then instantaneously reversing to forward while keeping the same speed.

1.5 Getting the position from the velocity

Since the instantaneous velocity v is the derivative of the position x with respect to time, we can get the position from the velocity by integrating with respect to time:

$$v(t) = \frac{dx}{dt}(t) \implies x(t) = C + \int v(t)\,dt$$

C is the integration constant. To find it, we need to know the position at one specific time. This is called the "initial condition". Let's say we're told that the position at some time t_0 is x_0, then we can write

$$x_0 = x(t_0) = C + \left[\int v(t)\,dt\right]_{t_0}$$

and solve for C (the brackets with t_0 subscript mean "compute the integral of x(t), which is itself a function of t, and evaluate it at $t = t_0$").

Problem 5: 1D velocity integration.

Apply the integration method above to a uniform motion with velocity v_0 and initial condition $x(0) = x_0$.

1. The velocity is v_0, which is a constant (does not depend on time). Integrate it with respect to time to get $x(t)$ as a function of v_0, t, and the integration constant C.
2. Plug the formula you obtained into the initial condition $x(0) = x_0$ and solve for C to get $x(t)$ as a function of v_0, t, and x_0.

Problem 6: Another 1D velocity integration.

Do the same with $v(t) = -t$ and $x(1) = 1.5$.

1.6 Acceleration

Just like the velocity is the (average or instantaneous) rate of change of the position, the acceleration is the (average or instantaneous) rate of change of the velocity. Mathematically:

Average acceleration between times t_1 and t_2: $\bar{a}(t_1, t_2) = \dfrac{v(t_2) - v(t_1)}{t_2 - t_1}$.

Instantaneous acceleration at time t: $a(t) = \lim\limits_{\Delta t \to 0} \bar{a}(t, t + \Delta t) = \lim\limits_{\Delta t \to 0} \dfrac{v(t + \Delta t) - v(t)}{\Delta t} = \dfrac{dv}{dt}$.

Note: Both the average acceleration and the instantaneous acceleration are defined in terms of the instantaneous velocity.

Since the instantaneous acceleration is the derivative of the instantaneous velocity with respect to time, and the instantaneous velocity is the derivative of the position with respect to time, the instantaneous acceleration is also the second derivative of the position with respect to time:

$$a(t) = \frac{dv}{dt} \text{ and } v(t) = \frac{dx}{dt} \implies a(t) = \frac{d^2x}{dt^2}.$$

Problem 7: Problem 2 continued.

1. Compute the instantaneous acceleration of the motion introduced in problem 2.
2. The sign of the acceleration can be read directly from the graph. Explain how, and show that it is consistent with what you obtained in question 1.

Problem 8: Sketching the acceleration.

For each motion from problem 4, sketch the acceleration below the position and velocity.

In motion 2, does the sign of $a(t)$ in each part of the motion match your expectation for something that slows down, stops, then speeds up? Explain.

1.7 Rates of change

The velocity and the acceleration are examples of rates of change. Consider a quantity x whose value depends on time. I'm using the same letter we used for the position, but doesn't have to be a position, it can be anything that varies through time: a position, a velocity, the distance between two points, and angle, the size of something, a temperature, a voltage, a current, a light intensity, a magnetic field, the energy of something, etc. Regardless of what x is, as long as it depends on time, i.e., whenever we can write it as a function of time $x(t)$, we can define its average and instantaneous rates of change:

Average rate of change between times t_1 and t_2: $\dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$.

Instantaneous rate of change at time t: $\lim\limits_{\Delta t \to 0} \dfrac{x(t + \Delta t) - x(t)}{\Delta t} = \dfrac{dx}{dt}$.

1.8 Getting the velocity and/or the position from the acceleration

Just like you can integrate $v(t)$ to get $x(t)$ (because $v = dx/dt$, see section 1.5), you can integrate $a(t)$ to get $v(t)$ (because $a = dv/dt$). Again, you need an initial condition (the value of $v(t)$ at a specific time) to solve for the integration constant.

If you have $a(t)$ and you want $x(t)$, you need to integrate twice. Two integration constants appear, one for each integration. To find their values, you need two initial conditions, for example the initial velocity and the initial position.

Relevance: The central result of mechanics – Newton's second law – allows us to predict the acceleration $a(t)$, however what we're typically interested in is predicting where an object will be in the future (its position), or how fast it will go (it velocity). Therefore, using the laws of mechanics often involves integrating the acceleration twice (once to get the velocity, and a second time to get the position).

The fact that you need two initial conditions, one for the velocity and one for the position, is also meaningful. Say you have a movie of a falling ball. As we'll see, the motion of falling objects is among the easiest ones to predict using the laws of mechanics. However, one frame is not enough to predict what will happen in the other frames. One frame gives you the position of the object (an initial position). With two frames however, you can compute the velocity of the object between the two frames and use it as the initial velocity. Then you can use the laws of mechanics to make a prediction.

1.9 Uniformly accelerated motion

This is a type of motion in which the acceleration is constant. Rather than postulate the velocity and position as a function of time, let's infer them from the acceleration as outlined in section 1.8.

Problem 9: Uniformly accelerated motion.

Consider a uniformly accelerated motion with (constant) acceleration $a(t) \equiv a_0$, initial velocity $v(0) \equiv v_0$, and initial position $x(0) \equiv x_0$.

1. Integrate the acceleration with respect to time to get the velocity as a function of time. Use the initial condition on the velocity to solve for the integration constant.
2. Integrate the velocity you just obtained with respect to time to get the position. Use the initial condition on the position to solve for the integration constant.

Problem 10: Braking distance.

At time $t = 0$, the front of the car is at $x = 0$. It is moving to the right with initial velocity v_0. It is decelerating with constant acceleration a_0.

1. What are the signs of v_0 and a_0?
2. Compute $v(t)$ as a function of v_0, a_0, and t. Sketch it from $t = 0$ until v reaches 0. How long does it takes the car to stop?
3. Compute $x(t)$ as a function of v_0, a_0, and t.
4. The braking distance d_B is the distance traveled over the course of the deceleration. Compute it as a function of v_0 and a_0.
5. What happens to d_B when v_0 is multiplied by two? First state the result mathematically, then in plain English.
6. Say you're going 80 mph instead of 60 mph. How many times faster is that? How many times longer is the braking distance?

Problem 11: Stopping distance.

The braking distance we computed in the previous problem is measured from the place where you start hitting the brakes. A more relevant distance to compute is the one traveled between the time you notice the need to brake and the time you come to a full stop. In this problem, the car starts off at $t = 0$ and $x = 0$ with velocity v_0, however the driver only starts braking at $t = t_R$ where t_R is the reaction time. Between $t = 0$ and $t = t_R$, the car maintains a constant velocity v_0. After $t = t_R$, the acceleration is a_0, which is constant and negative like in problem 10.

1. What is the acceleration when $0 < t < t_R$?
2. First we focus on the reaction phase ($0 < t < t_R$), when the velocity is v_0. Integrate the velocity and use the initial condition to obtain $x(t)$.
3. Now we focus on the braking phase ($t > t_R$). The initial conditions for the braking phase are that x and v have the same values they had at the end of the reaction phase. Write those initial conditions in terms of v_0 and t_R.
4. Integrate the acceleration and use the initial conditions from question 3 to get $v(t)$ during the braking phase.
5. Integrate the $v(t)$ from question 4 and use the initial conditions from question 3 to get $x(t)$ during the braking phase.
   Hint: If your answer contains $\frac{1}{2}a_0t^2 - a_0t.t_R + \frac{1}{2}a_0t_R^2$, use the square-of-a-difference formula, $(a-b)^2 = a^2 - 2a.b + b^2$, to simplify it. It will make question 6 much easier.
6. Compute the total distance traveled from the time the driver notices the need to stop ($t = 0$) to the time the car stops ($v = 0$).
7. In retrospect, the final result could have been predicted without integration (but using the result of problem 10). Explain how, and why it's a legitimate argument.

Problem 12: Worked out problems from the book.

See examples 3.10, 3.11, 3.12, and 3.13 from University Physics Volume 1, section 3.4. Example 3.10 is similar to problem 11 above.

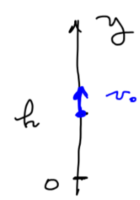
1.10 Free fall

A free fall is a motion under the sole influence of gravity. Falling objects typically experience air friction as well, however in a lot of situations it is reasonable to neglect it (example: jumping from a bench; counter-example: jumping from a plane). On the surface of the Earth, all free fall motions have the same constant acceleration: $g = 9.81\,\mathrm{m/s^2}$, pointing down.

In problems, it is usually best to keep g as a literal until you have numerical values for everything else in the formula.

Problem 13: Vertical ball throw.

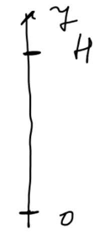
An object is thrown up at $t = 0$ from a height h with upward speed v_0.



1. Compute $v(t)$, then $y(t)$.
2. Compute the time t_1 at which y is equal to h again.
   Note: You will find two times at which $y = h$; the initial time, and the one we're after.
3. Compute $v(t_1)$. What's remarkable about it?
4. Compute the time t_2 at which the ball hits the ground as a function of h, v_0, and g.
   Note: You will find two times, one negative and one positive. Since the ball wasn't in free fall before $t = 0$, we want the positive one.
5. Compute the velocity of the ball as it hits the ground when $v_0 = 1\,\mathrm{m/s}$ and $h = 1\,\mathrm{m}$.

Problem 14: Waterfall clock.

This problem was inspired by this video. It made me wonder how one would predict and control the way the symbols stretch as they fall. I made some simplifying assumptions to keep calculations manageable.



A stream of water is released from a height H with no initial velocity, starting at $t = 0$ and ending at $t = T$. The first droplet, released at $t = 0$, becomes the bottom end of the stream. The last droplet, released at $t = T$, becomes the top of the stream. We're going to assume each of those two droplets experiences a free fall from the moment it's released to the moment it hits the ground ($y = 0$). We're not going to study the rest of the water; we'll just assume that it occupies the space between the top and bottom droplets. Let y_{top} be the y coordinate of the top droplet and y_{bot} that of the bottom droplet.

1. Integrate the free fall acceleration twice and use the initial conditions to compute $y_{\text{top}}(t)$ and $y_{\text{bot}}(t)$.
2. Compute the vertical size $h(t)$ of the stream (distance from top to bottom).
3. Let h_1 be the initial size of the stream, measured when the top of it is released and the stream becomes whole. How must we choose T to get $h_1 = 0.5\,\text{m}$?
4. Let h_2 be the final size of the stream, measured as the bottom of the stream reaches the ground. Compute h_2 as a function of the parameters of the problem.
5. Say the stream is dropped from a height $H = 2\,\text{m}$, and we want an initial stream height $h_1 = 0.5\,\text{m}$ (same as question 3). When does the stream's bottom touch the ground. What is its height then?

1.11 Relative velocity

We've talked about the position of an object relative to the other. Imagine two objects, one at point A and one at point B. As discussed in the *Foundations* lecture notes (section *Relative position vectors*), the position of object B relative to object A is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, whose coordinate is $x_{AB} = x_B - x_A$. Now imagine both objects are moving, i.e., x_A and x_B are actually functions of time.

The instantaneous velocity of A is defined as $v_A(t) = \frac{d}{dt}\Big(x_A(t)\Big)$. For B is $v_B(t) = \frac{d}{dt}\Big(x_B(t)\Big)$. The instantaneous velocity of B relative to A is defined the same way, but using the relative position coordinate:

$$v_{AB}(t) \equiv \frac{d}{dt}\Big(x_{AB}(t)\Big) = \frac{d}{dt}\Big(x_B(t) - x_A(t)\Big).$$

As an immediate consequence, we can also write:

$$v_{AB}(t) = v_B(t) - v_A(t).$$

In words: the velocity of B relative to A is the velocity of B minus the velocity of A.

We can do the same with the average velocity between two times t_1 and t_2:

$$\overline{v}_{AB}(t_1, t_2) \equiv \frac{x_{AB}(t_2) - x_{AB}(t_1)}{t_2 - t_1}.$$

Note

The regular velocity is also a relative velocity: it's the velocity relative to the origin O.

Problem 15: Relative average velocity.

Show that, just like the instantaneous velocity, the average velocity of B relative to A is equal to the average velocity of B minus the average velocity of A. In other words, show that $\overline{v}_{AB}(t_1, t_2) = \overline{v}_B(t_1, t_2) - \overline{v}_A(t_1, t_2)$.

1.12 Relative acceleration

The logical next step is to define the instantaneous and the average accelerations of B relative to A as

$$a_{AB}(t) \equiv \frac{d}{dt}\Big(v_{AB}(t)\Big), \quad \overline{a}_{AB}(t_1, t_2) \equiv \frac{v_{AB}(t_2) - v_{AB}(t_1)}{t_2 - t_1}.$$

As expected, they obey $a_{AB}(t) = a_B(t) - a_A(t)$ and $\overline{a}_{AB}(t_1, t_2) = \overline{a}_B(t_1, t_2) - \overline{a}_A(t_1, t_2)$.

Problem 16: Waterfall clock – Motion of the bottom relative to the top.

This is a continuation of problem 14.

1. Use your results for v_{top} and v_{bot} to compute the instantaneous velocity of the bottom relative to the top as a function of time. How does it depend on time? What does that tell you about the way the stream stretches?
2. Compute the instantaneous acceleration of the bottom of the stream relative to the top of the stream. How does it relate to the relative velocity from question 1?
3. What is the mathematical relationship between the relative velocity from question 1 and the height $h(t)$ of the stream? Use your results from problem 14 to show that this relationship is indeed verified.

1.13 Velocity and acceleration as predictors of future positions

Imagine you have the position of an object as a function of time, $x(t)$, up to a time t_0, and you want to predict what will happen after that.

If you have no other information, it's reasonable to assume that x is going to keep changing at the same rate. Graphically, that corresponds to extending the curve with a straight line that has the same slope $x(t)$ had right as it reached t_0. That straight line is the tangent to the curve at t_0. Its equation is $x_{\text{pred}}(t) = x(t_0) + v(t_0).(t - t_0)$. If we plug a time t larger than t_0 into this equation, it will tell us where we can expect the object to be at time t_0 under the assumption that its position keeps changing at the same rate.

Another way to think about this is "constant rate" approximation is to say that we assume the motion after t_0 will be uniform. The initial information for that uniform motion is that it starts at $t = t_0$, $x = x(t_0)$ (the last known position of the actual motion), with velocity $v(t_0)$ (the last known velocity of the actual motion). Integrating this constant velocity and using the initial condition is another way to obtain the equation of the tangent: $x(t) = x(t_0) + v(t_0).(t - t_0)$.

If we want to refine that prediction, we may assume that, after t_0, the curve keeps not only the same slope but also the same curvature. In mathematical terms, this new predicted curve starts at t_0 with $x = x(t_0)$, $dx/dt = v(t_0)$, and $d^2x/dt^2 = a(t_0)$, and d^2x/dt^2 remains constant after that (constant curvature). We can get the equation of that predicted curve by integrating twice and using the initial conditions, which yields $x_{\text{pred}}(t) = x(t_0) + v(t_0).(t - t_0) + a(t_0).(t - t_0)^2/2$.

Both the prediction formulas above are examples of Taylor expansions, which you may hear about in Calc 2. The first one (constant velocity) is a first-order Taylor expansion around $t = t_0$. The second one (constant curvature) is a second-order Taylor expansion around $t = t_0$.

2 2D motion

Virtually every quantity we defined for 1D motion can be generalized to 2D, however many of those quantities are now vectors.

2.1 Position vs time

We now need two coordinates to locate our moving object: $x(t)$ and $y(t)$. The position vector is $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. The trajectory of the object now needs to be graphed in the xy plane, while graphing the position vs time now requires two graphs: $x(t)$ and $y(t)$.

2.2 Distance traveled

We touched upon the distance traveled in the fish activity. We mentioned that, in order to get the full distance traveled, we need to break the motion down into segment during which the direction of motion did not change, compute the length of each segment, and add them up.

When the trajectory is curved, this turns out to be quite difficult. The direction of motion changes literally all the time, so the motion needs to be broken down into an infinity of infinitely short segments. All of their lengths then need to be computed and added up. This can be done with an integral, but we'll stick to situations where this is not necessary, i.e., situations in which the trajectory is simple enough that we don't need the integral.

2.3 Velocity

The average velocity is

$$\vec{v}(t_1,t_2) \equiv \frac{\text{displacement vector}}{\text{time elapsed}} \equiv \frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1} = \begin{bmatrix} \dfrac{x(t_2)-x(t_1)}{t_2-t_1} \\ \\ \dfrac{y(t_2)-y(t_1)}{t_2-t_1} \end{bmatrix}$$

It is a vector. Its x component is the average velocity you would compute if this was a 1D motion with coordinate $x(t)$. Its y component is the average velocity you would compute if this was a 1D motion with coordinate $y(t)$. $\vec{v}(t_1,t_2)$ is parallel to the displacement vector $\vec{r}(t_2)-\vec{r}(t_1)$, which goes from the position of the object at t_1 to its position at t_2. It always points in the forward direction of the motion (see Exam 1, problem 3 for the breakdown of what happens when $t_2 < t_1$ vs $t_2 > t_1$).

The instantaneous velocity is the limit of the average velocity when the two times become infinitely close, which is also the derivative of the position vector with respect to time:

$$\vec{v}(t) \equiv \lim_{\Delta t\to 0} \vec{v}(t, t+\Delta t) \equiv \frac{d\vec{r}}{dt}(t)$$

Again, it is a vector, its x component is the instantaneous velocity you would compute for a 1D motion with coordinate $x(t)$, and its y component is the instantaneous velocity you would compute for a 1D motion with coordinate $y(t)$. In particular:

$$\vec{v}(t) = \begin{bmatrix} \dfrac{dx}{dt}(t) \\ \\ \dfrac{dy}{dt}(t) \end{bmatrix}$$

Note: Deriving and integrating vectors

As long as the basis vector are constant, deriving a vector is the same thing as deriving each of its components independently, and integrating a vector is the same thing as integrating each of its components independently.

However, this stops being true when the basis vectors are not constant, specifically when they depend on the variable you're deriving or integrating with respect to. This arises for example when using the polar basis in a circular motion.

2.4 Speed

As discussed in the context of 1D motion, the speed is the distance traveled divided by the time elapsed. Like the velocity, it can be average or instantaneous. Like the distance traveled, it can be difficult to compute when the trajectory is curved.

If the direction of motion is constant, however (straight line traveled in a consistent direction), the distance traveled is simply the magnitude of the displacement vector, therefore the speed is the magnitude of the velocity vector:

$$\begin{aligned} \left\|\vec{v}(t_1,t_2)\right\| &= \left\|\frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1}\right\| = \frac{\left\|\vec{r}(t_2)-\vec{r}(t_1)\right\|}{|t_2-t_1|} \\ &= \frac{\text{distance traveled between } t_1 \text{ and } t_2}{\text{time elapsed}} \\ &\equiv \text{average speed between } t_1 \text{ and } t_2 \end{aligned}$$

A similar argument applies to the instantaneous speed. Even if the direction of motion is not constant, it can be treated as constant between t_1 and t_2 in the limit $t_2 \to t_1$ because it hasn't had enough time to change in any significant way. The end result is the same as above: the (instantaneous) speed is the magnitude of the (instantaneous) velocity.

$$\begin{aligned}\|\vec{v}(t)\| &\equiv \left\|\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}\right\| = \lim_{\Delta t\to 0}\frac{\|\vec{r}(t+\Delta t)-\vec{r}(t)\|}{|\Delta t|}\\ &= \lim_{\Delta t\to 0}\frac{\text{distance traveled between } t \text{ et } t+\Delta t}{\text{time elapsed}}\\ &\equiv \text{instantaneous speed at } t\end{aligned}$$

Problem 17: Fish activity – Part 6.

1. Compute the average velocity of fish 1 between your row and the next.
2. Compute the corresponding speed, assuming the motion is straight between two consecutive rows.

2.5 Uniform motion

As in 1D, a uniform motion is a motion with constant velocity (both average and instantaneous). In 2D, that means both components of the velocity vector must remain constant. In other words, both the speed *and* and the direction of motion must remain constant.

Problem 18: Integrating a uniform motion.

An object has constant velocity $\vec{v}_0 = \begin{bmatrix} v_{0x} \\ v_{0y} \end{bmatrix}$ and initial position $\vec{r}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$. Integrate the velocity and use the initial condition to obtain its position $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ as a function of time.

Problem 19: Fish activity – Part 7.

Imagine the row below yours is the last one in the spreadsheet. In the absence of any information about how the fish may change their motion after that last available row, it's reasonable to assume that, at least for a little bit, they're going to keep moving at the same speed in the same direction, i.e., with the same velocity, i.e., their motion is uniform after the last available row. It's not exactly true, and the more time passes after the last row, the less likely it is to remain true, but it's probably reasonable over a couple of seconds, and it's the best we can do absent additional information.

1. Use the pretend last known position (row below yours) and the pretend velocity computed between your row and the next row to predict the position of fish 1 as a function of two rows down from your row. *Hint: Treat this as a uniform motion with constant velocity $\vec{v}_0$ equal to the velocity you computed in problem 17, initial condition $\vec{r}(t_{i+1}) = \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix}$, and integrate like you did in problem 18.*
2. Check your prediction against the actual position two rows down. Explain why they're not exactly the same using words and a sketch of the fish's trajectory.
3. Use the same assumptions to predict the position 0.5 seconds after your row.
4. Since there are 0.33 seconds between two consecutive rows, 0.5 seconds after your row is some time between the row below yours (which is 0.33 seconds after your row) and the one below that (which is 0.66 seconds after your row). We don't know the exact position at that time, but we can get estimate it using linear interpolation between the two nearest row, i.e., assuming the motion is uniform (straight line at constant speed) between any two consecutive rows.

2.6 Acceleration

The average acceleration is

$$\vec{a}(t_1, t_2) \equiv \frac{\text{displacement vector}}{\text{time elapsed}} \equiv \frac{\vec{v}(t_2) - \vec{v}(t_1)}{t_2 - t_1} = \begin{bmatrix} \dfrac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} \\ \\ \dfrac{v_y(t_2) - v_y(t_1)}{t_2 - t_1} \end{bmatrix}$$

It is a vector.

The instantaneous acceleration is the limit of the average acceleration when the two times become infinitely close, which is also the derivative of the instantaneous velocity vector with respect to time, and the second derivative of the position vector with respect to time:

$$\vec{a}(t) \equiv \lim_{\Delta t \to 0} \vec{a}(t, t + \Delta t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}(t)$$

In terms of components:

$$\vec{a}(t) = \begin{bmatrix} \dfrac{dv_x}{dt} \\ \\ \dfrac{dv_y}{dt} \end{bmatrix} = \frac{d^2\vec{r}}{dt^2}(t) = \begin{bmatrix} \dfrac{d^2x}{dt^2} \\ \\ \dfrac{d^2y}{dt^2} \end{bmatrix}$$

where v_x and v_y are the components of the instantaneous velocity vector. Its x component is the instantaneous acceleration you would compute for a 1D motion with coordinate $x(t)$, and its y component is the instantaneous acceleration you would compute for a 1D motion with coordinate $y(t)$.

2.7 Uniformly accelerated motion. Free fall

Relevant section in the book: 4.3.

A uniformly accelerated motion is a motion with constant acceleration (both average and instantaneous). In 2D, that means both components of the acceleration vector must remain constant. In other words, both the magnitude and the direction of acceleration vector must remain constant.

Free fall

A prime example of uniformly accelerated motion is the free fall motion, during which the acceleration is equal to the acceleration of gravity regardless of the nature of the falling object. The acceleration-of-gravity vector on the surface of the Earth is noted $\vec{g}$. Its magnitude is $g = 9.81\,\mathrm{m\,s^{-2}}$, and it points down. In a coordinate system with the x axis horizontal and the y axis vertical and pointing up, the components of $\vec{g}$ are $\begin{bmatrix} 0 \\ -g \end{bmatrix}$. This won't always be the case though; sometimes we'll use tilted coordinate system, for example when we talk about an object sliding down a tilted plane.

Note: This type of motion is sometimes called *projectile motion*. It's not exactly synonymous with *free fall*, but for the purpose of what we're doing here it might as well be.

Problem 20: 2D free fall.

An object is thrown at time $t = 0$ from a position $\vec{r} = \vec{0}$ with initial velocity $\vec{v}(0) = \begin{bmatrix} v_{0x} \\ v_{0y} \end{bmatrix}$. Both v_{0x} and v_{0y} are positive. We use the "standard" x and y axes: x is horizontal, y is vertical, pointing up, and $y = 0$ represents the ground. We neglect air friction, so the object is free fall, meaning the acceleration is constant and equal to the acceleration of gravity: $\vec{a} = \vec{g} = -g\hat{y}$.
At the end of every question, check the dimensional consistency of your result.

1. Integrate $\vec{a}$ to get $\vec{v}(t)$ (or its components $v_x(t)$ and $v_y(t)$, it's equivalent).
2. Integrate $\vec{v}(t)$ to get $\vec{r}(t)$ (or its components $x(t)$ and $y(t)$, it's equivalent).
3. Compute the time t_1 at which the object is the highest. Then, compute its position at that time. What is the maximum height H?
4. Compute the time t_2 at which the object hits the ground. Then, compute its position at that time. How far did the object reach, i.e., what is the (horizontal) distance D between the initial position and the landing position?
5. Sketch each component of the acceleration, the velocity, and the position from $t = 0$ to $t = t_2$.
6. Write y as a function of x and the initial data (g, $\vec{v}_0$, but not t). Sketch the trajectory. What type of curve is it? *Hint: To get $y(x)$, first invert $x(t)$ to get $t(x)$, then plug that into $y(t)$.*
7. The highest height reached (H) and the distance traveled before hitting the ground (D) can also be obtained from the equation of the trajectory ($y(x)$). Do it.

Problem 21: Optimal throw.

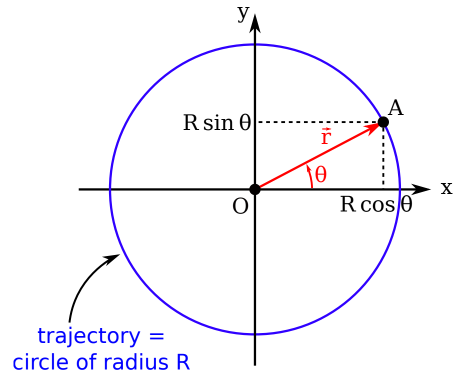
This is a continuation of problem 20. The key difference is that we're going to write results in terms of the initial velocity's magnitude v_0 and angle with the horizontal θ_0 rather than in terms of its cartesian components v_{0x} and v_{0y}. That will allow us to ask the following question: assuming we have to throw at a specific speed (which could be the maximum speed we're able to throw at), what is the best way to pick the angle θ_0 to maximize the range of the throw (how far it lands)?

1. Sketch the initial velocity. Include the components v_{0x} and v_{0y}, the magnitude v_0, and the angle θ_0 with the x axis on the sketch. Then, write v_{0x} and v_{0y} in terms of v_0 and θ_0.
2. Rewrite the distance D at which the object lands, which you computed in problem 20, in terms of v_0 and θ_0 rather than v_{0x} and v_{0y} (in other words, eliminate v_{0x} and v_{0y} in favor of v_0 and θ_0).
3. We now look for the value of θ_0 that maximizes the distance traveled before hitting the ground. In other words, we look for the maximum of D as a function of θ_0 assuming everything else (g, v_0) is constant.
   (a) Both v_{0x} and v_{0y} are positive. What does that imply for θ_0?
   (b) Use calculus to find the value of θ_0 that maximizes D. That's the optimal throwing angle if you're going for range. You should only find one solution in the relevant quadrant. Then, compute the corresponding range D.
   (c) We're going to check the maximum we found by graphing $D(\theta_0)$. First, explain why the location of the maximum does not depend on the values of v_0 and g (as long as they are positive, which they are).
   (d) Since the values of v_0 and g do not matter, we can set them to any (positive) value. Let's use 1 for both. Graph $D(\theta_0)$ using Desmos' graphing calculator. The extrema should show up as grey dots. Find the one we're after.

Note: University Physics Volume 1 Section 4.3 links to an interactive animation illustrating the role of the throwing speed and angle.

2.8 Circular motion

An object following a circular trajectory is said to be performing a *circular motion.*



2.8.1 Position

The motion is easiest to describe in a coordinate system centered on the center of the circle. The position vector is $\vec{r}(t) = \begin{bmatrix} R\cos\theta(t) \\ R\sin\theta(t) \end{bmatrix}$. The radius R is constant. That's what makes this a circular motion. The angle θ, on the other hand, is a function of time. If it was constant too, there would be no motion at all.

2.8.2 Angular velocity

Since R remains constant throughout, keeping track of the motion is really about keeping track of the variations of θ. In a way, you can think of the circle as a curved axis, and the circular motion as a "1D" motion along that curved axis. You can then think of θ as a sort of coordinate that tells you where you are along that curved axis. It's not quite the same as a regular position coordinate – for one, it is dimensionless where position coordinates ahave dimension of a length –, but it fulfills a similar role.

It follows that the time derivative of $\theta(t)$ plays a role similar to the velocity of a 1D motion. Its actual name is the angular velocity, and it is often noted with the greek letter *omega* (ω):

$$\text{Angular velocity:} \quad \omega(t) \equiv \frac{d\theta}{dt}(t)$$

2.8.3 Velocity

Problem 22: Velocity in a circular motion.

1. Use $\vec{r}(t) = \begin{bmatrix} R\cos\theta(t) \\ R\sin\theta(t) \end{bmatrix}$ to compute the instantaneous velocity vector $\vec{v}$ as a function of R, $\theta(t)$, and $\omega(t)$.
2. Compute the instantaneous speed v as a function of the same variables. Once you've fully simplified it, you should obtain a relationship between v, R, and ω.
3. Compute the radial and tangential components of the velocity. Explain why the result was largely predictable.

2.9 Uniform circular motion

If the speed of a circular motion is constant (doesn't depend on time), the circular motion is said to be *uniform.* This is not to be confused with *uniform motion*, which means constant velocity (constant speed *and* constant direction).

Problem 23: Velocity in a uniform circular motion.

1. What constraint does the speed being constant imply for ω? What does that imply for θ?
2. Assuming the initial angle is $\theta(0) = \theta_0$, compute θ as a function of t, ω, and θ_0.
3. Assuming $\theta_0 = 0$, sketch $\theta(t)$, $x(t)$, and $y(t)$ one below the other.

Problem 24: Acceleration in a uniform circular motion.

1. Use the velocity vector you obtained in problem 22 to show that $\vec{a}(t) = -\omega^2\vec{r}(t) = -\frac{v^2}{R}\hat{r}(t)$ at all times during a uniform circular motion. Note that ω is now a constant, whereas in problem 22 ω could depend on time (because we were considering a general circular motion whereas we're now focusing on a uniform circular motion).

2. The acceleration vector is defined as $\vec{a} = \lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}$. Assuming Δt is positive, $\vec{a}$ has the same direction as $\vec{v}(t+\Delta t)-\vec{v}(t)$.
Sketch the trajectory. Pick a point on it. Sketch the point and its velocity vector (assume the motion is counterclockwise). Sketch another point corresponding the the position a little bit later (not too far from the first point, but not so close that the sketch becomes cluttered). Sketch the velocity vector of that new point. Use a graphical construction to obtain the difference between the second velocity vector and the first one. Which direction does it point? In light of this graphical construction, explain why it makes sense that $\vec{a}$ is along $-\hat{r}$, as we showed in question 1.

Uniform circular motion as a periodic motion

Uniform circular motion is periodic. Every time the angle $\theta(t) = \theta_0 + \omega t$ increases by 2π, every quantity which only depends on time through a sine or cosine of θ (including the position $\vec{r}$, the velocity $\vec{v}$, the acceleration $\vec{a}$, the polar basis vectors $\hat{r}$ and $\hat{t}$) is a periodic function of time.

Problem 25: Periodic motion.

1. Compute the time T it takes for θ to increase by 2π as a function of ω. T is known as the period of the motion.
2. Show that the position is indeed a periodic function of time with period T, i.e., show that $\vec{r}(t+T) = \vec{r}(t)$ for any t.
3. How much distance does the object travel over one period (one full rotation)? Use this result to write the average speed v as a function of R and T. Then, use the result of question 1 to get a relationship between v, R, and ω.

2.10 Velocity and acceleration vectors: Graphical approach

The velocity vector $\vec{v}$ is the rate of change of the position vector $\vec{r}$. It tells us how the position is changing. If we know the position $\vec{r}(t)$ and the velocity $\vec{v}(t)$, we can estimate the position at a later time $t+\Delta t$ as

$$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \Delta t.\vec{v}(t)$$

This is the 2D version of what we discussed in section 1.13. Graphically, the position $\vec{r}(t+\Delta t)$ is obtained by starting at the position $\vec{r}(t)$, then moving along the direction of $\vec{v}$ by a distance proportional to Δt (the more time passes, the more the object moves along the direction of $\vec{v}$).

The same reasoning can be applied to the acceleration and the velocity. The acceleration vector $\vec{a}$ is the rate of change of the velocity vector $\vec{v}$. It tells us how the velocity is changing. If we know the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$, we can estimate the velocity at a later time $t+\Delta t$ as

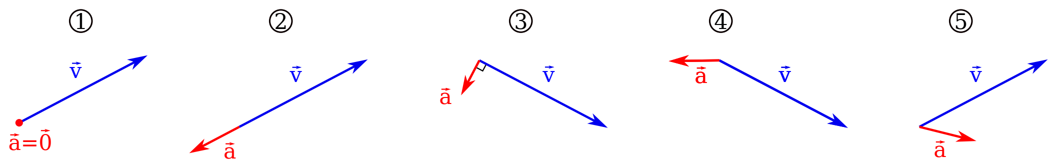
$$\vec{v}(t+\Delta t) \approx \vec{v}(t) + \Delta t.\vec{a}(t)$$

Graphically, the velocity $\vec{v}(t+\Delta t)$ is obtained by starting with the velocity $\vec{v}(t)$ and graphically adding a vector that has the same direction as $\vec{a}(t)$ but whose length is proportional to Δt.

Problem 26: Velocity and acceleration: graphical approach.

In each of the four situations below, you are given $\vec{v}(t)$ and $\vec{a}(t)$. The goal is to graphically construct the approximate the position and the velocity at $t+\Delta t$, then the position at $t+2\Delta t$, then sketch the trajectory and describe the motion in words.

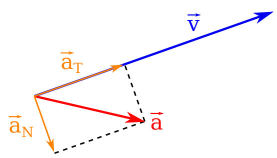
Assume that $\vec{v}.\Delta t$ is 1.5 times the length of $\vec{v}$, and that $\vec{a}.\Delta t$ is the same length as $\vec{a}$. The exact values are not critical, this is just to keep the sketches readable.



2.11 Tangential and normal accelerations: Graphical Approach

What we started to see in the last problem is that when the acceleration is parallel to the velocity, it changes the speed but not the direction. Conversely, when the acceleration is perpendicular to the velocity, it changes the direction but not the speed (as long as Δt is very small).

More generally, any acceleration vector $\vec{a}$ can be decomposed as $\vec{a}_T + \vec{a}_N$ where $\vec{a}_T$, known as the *tangential acceleration*, is parallel to $\vec{v}$, and $\vec{a}_N$, known as the *normal acceleration*, is perpendicular to $\vec{v}$. You can think of this as defining a cartesian basis whose first unit vector is along $\vec{v}$ (that would be $\hat{v}$), then projecting $\vec{a}$ onto that basis.



$$\vec{a} = \vec{a}_T + \vec{a}_N$$
$$\vec{a}_T \parallel \vec{v}$$
$$\vec{a}_N \perp \vec{v}$$

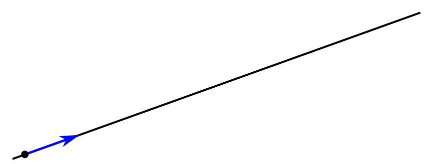
The reason why that decomposition of is important is because each of $\vec{a}_T$ and $\vec{a}_N$ controls a distinct aspect of the motion:

- $\vec{a}_T$ controls the rate of the change of the speed.
- $\vec{a}_N$ controls the rate of change of the direction.

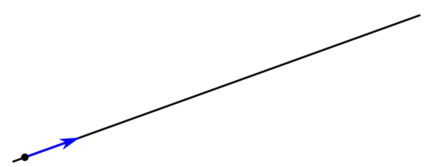
Problem 27: Velocity and acceleration: graphical approach 2.

In each of the following situations, sketch the instantaneous velocity and the instantaneous acceleration at a few equally spaced times (t, $t+\Delta t$, $t+2\Delta$, etc) along the motion. The black curve represents the trajectory. The black dot represents the initial position. The blue arrow represents the initial velocity.

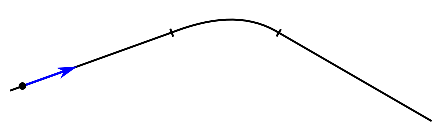
1. Straight motion, constant speed.



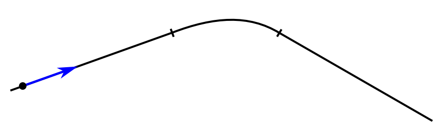
2. Straight motion, speeding up.



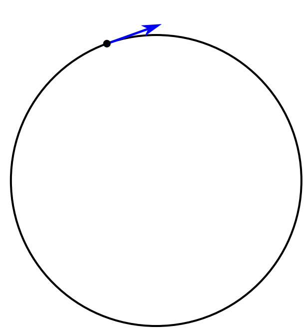
3. Straight, then turn, then straight again, all at constant speed. The small transverse lines show where the trajectory starts curving and were it starts being straight again.



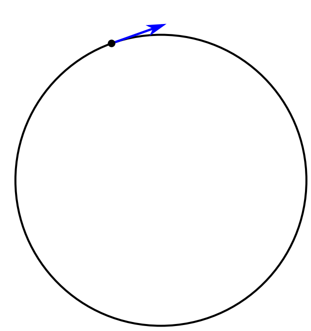
4. Same, but speeding up the whole time.



5. Uniform circular motion.



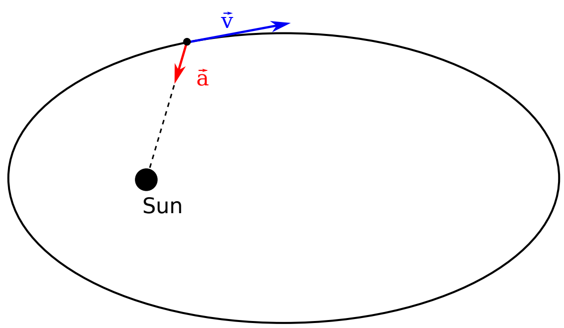
6. Circular motion, speeding up.



Problem 28: Velocity and acceleration: graphical approach 3.

The sketch shows the orbit of a comet around the sun (this really applies to any orbit, it's just that the point this problem tries to make is easier to see if the orbit is elongated, and comets often have elongated orbits). The orbit is an ellipse with the Sun as one of its focal points. The acceleration always points toward the Sun (we'll discuss that more later in the course; in short, the acceleration is controlled by the gravitational force exerted by the Sun on the comet, and the direction of that attractive force is toward the Sun).

1. Sketch the comet, its velocity, and its acceleration at a few locations along its orbit.
2. Indicate the part of the orbit over which the comet speeds up, and the part over which it slows down.



Problem 29: Bouncing ball: graphical approach.

An object is thrown up vertically from the ground. While it is in the air, its motion is a free fall, i.e., its acceleration is the acceleration of gravity $\vec{g}$, which points down. When the object hits the ground, it experiences an instantaneous elastic collision, meaning the y component of its velocity vector is reversed (changes sign). During that brief time, it is no longer in free fall. After the bounce, the bounce is again in free fall, until it hits the ground again.

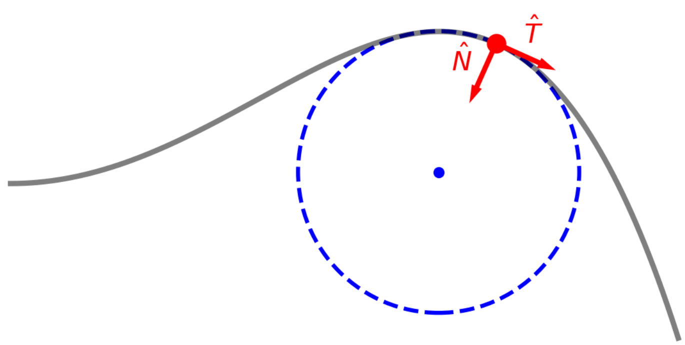
1. Sketch the initial situation (the ground, the object, and its velocity vector). Draw the acceleration on the sketch. What does the acceleration tell us about the way the velocity evolves during the free fall. Explain why that is consistent with intuition.
2. When the object reaches its highest point, its vertical velocity reaches zero (if you don't remember why, revisit previous free fall problems). For just an instant, it is like suspended mid-air. A common belief is that at that instant the object is not accelerating. If this was true ($\vec{a} = 0$), what would happen to $\vec{v}$ next? Why is that not consistent with reality? What actually happens to the acceleration at the apex?
3. Sketch the velocity vector right before the object hits the ground, then right after it hits the ground. What is the direction of the average acceleration during those two times (i.e., during the bounce)?
4. If the bounce is truly instantaneous, then the time interval between the two velocities above is $\Delta t = 0$. What does that imply for the acceleration? Does that seem realistic?

2.12 Tangential and normal acceleration: Quantitative Approach & Frenet Basis

The normal acceleration we found when we studied circular motion is much more general than it looks.

In a non-uniform circular motion, the normal acceleration is still $\vec{a}_N = -\frac{v^2}{R}\hat{r}$, except there is now also a tangential acceleration $\vec{a}_T = \frac{dv}{dt}\hat{v}$ where $v = \|\vec{v}\|$ is the speed and $\hat{v} = \vec{v}/v$ is the unit vector along the velocity. Since the motion is circular, $\hat{v}$ is either $\hat{t}$ (if the motion is counterclockwise) or $-\hat{t}$ (if the motion is clockwise). Finally, the full acceleration is the sum of the tangential and normal accelerations: $\vec{a} = \frac{dv}{dt}\hat{v} - \frac{v^2}{R}\hat{r}$

As it turns out, this formula applies to any motion, circular or not. We just need a slightly more general definition of R and $\vec{r}$. Let's think about a trajectory (a curve), and a point on that trajectory. The tangent to the trajectory at that point is the line that is closest to the curve at that point. In a similar way, we can define the *osculating circle* as the circle that best approximates (is closest to) the trajectory around the point:



The center of that circle is called the *center of curvature* of the trajectory at that point. The radius of the circle is called the *radius of curvature* of the trajectory at that point. In a circular trajectory, the osculating circle is just the trajectory. In a non-circular motion, the circle keeps changing as the point moves along the trajectory. There's an animated version of the figure on canvas (same page as the lecture notes; the video is called "frenet.mp4").

In order to write the acceleration in a general motion, we introduce the tangent unit vector $\hat{T}$, which is just another name for $\hat{v}$, and the normal unit vector $\hat{N}$, which is the unit vector pointing from the point to the center of curvature. In a circular motion, $\hat{N}$ is just $-\hat{r}$. With those notations, we can write the acceleration vector as:

$$\vec{a} = \frac{dv}{dt}\hat{T} + \frac{v^2}{R}\hat{N}$$

where v is the speed and R is the radius of curvature. When written like this, the formula applies to any motion, even non-uniform non-circular motions.

This form of the normal-tangential decomposition will come up repeatedly in the Mechanics chapter to discuss. For example, we'll use it to understand the centrifugal "force" felt in a turning car, the motion of a satellite around the Earth, and to introduce the concept of energy.

Problem 30: Revisiting normal/tangential acceleration.

$a_N = v^2/R$ tells us how the length of the normal acceleration depends on the motion's local speed and radius of curvature. What does it tell us about the acceleration in questions 5 and 6 of problem 27? Redraw the acceleration arrows to taking this new information into account.

2.13 Relative velocity

In section 1.11, we defined the relative velocity of a point B relative to another point A as the rate of change of the position of B relative to A: $v_{AB} = \frac{dx_{AB}}{dt}$ where $x_{AB} = x_B - x_A$. By plugging the definition of x_{AB} into the definition of v_{AB}, we then showed that $v_{AB} = v_B - v_A$, i.e., the velocity of B relative to A is equal to the velocity of B (relative to the origin) minus the velocity of A (relative to the origin).

The same definitions and relationships apply in 2D:

$$\vec{v}_{AB} = \frac{d\overrightarrow{AB}}{dt} = \vec{v}_B - \vec{v}_A.$$

As discussed in the 1D case, one way to picture $\vec{v}_{AB}$ is to imagine that we're filming A and B, but our camera follows A, i.e., we move the camera so as to keep A in the spot on the screen (while keeping the camera horizontal). If we watch the movie recorded this way, A will look like it's not moving, and B will look like it's moving at $\vec{v}_{AB}$.

Alternate notation

Writing the velocity of B relative to A as $\vec{v}_{AB}$ highlights the fact that it's the rate of change of $\overrightarrow{AB}$, the vector going from A to B. Some people, however, prefer to write to write the letters in the subscript in reverse order, $\vec{v}_{BA}$, because it then matches the order in which those letters appear in the phrase "velocity of B relative to A". In these lecture notes, $\vec{v}_{AB}$ will always mean "of B relative to A". When I feel the reverse order is beneficial, I will use a slightly different notation with a slash: $\vec{v}_{B/A}$. Here the slash stands for "relative to" in the phrase "velocity of B relative to A". This second notation is particularly useful when the two names of the two points are words rather than letters. For example, I may write the velocity of a person relative to the car they're in as $\vec{v}_{\text{person/car}}$. In this example, the other notation would be a little awkward, I would have to write something like $\vec{v}_{\text{CarPerson}}$ where CarPerson carries the idea that this is the rate of change of the vector that goes from the car to the person (the position of the person relative to the car). In summary, the relationship between the two notations is $\vec{v}_{B/A} = \vec{v}_{AB} = \dfrac{d\overrightarrow{AB}}{dt}$.

Problem 31: Fish activity – Part 8.

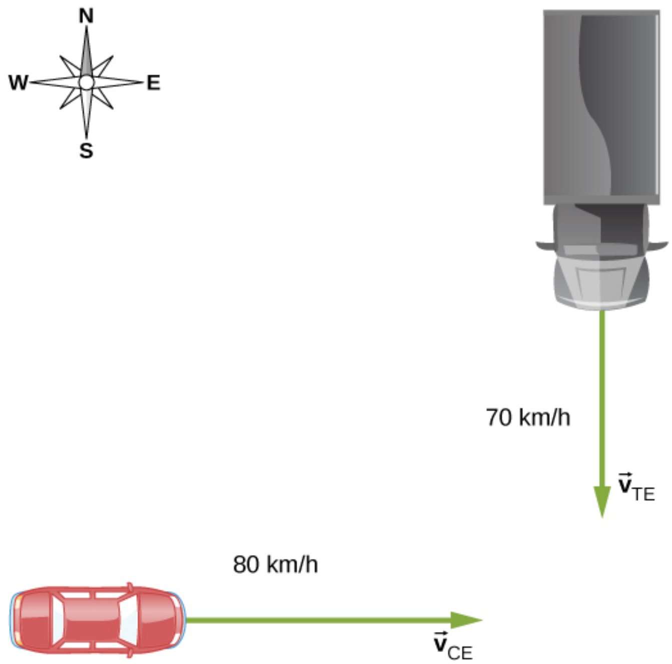
Compute the velocity of fish 2 relative to fish 1 between your row and the next. Compute it both ways: as the rate of change of the relative position vector, and as the difference between the velocity vector of fish 2 and that of fish 1. Make sure you get the same result both ways.

Problem 32: Examples 4.13 and 4.14 from the book.

From section 4.5 of University Physics Volume 1.

1. Example 4.13: Motion of a Car Relative to a Truck

   A truck is traveling south at a speed of 70 km/h toward an intersection. A car is traveling east toward the intersection at a speed of 80 km/h. What is the velocity of the car relative to the truck?



   Note: The solution given in the book computes the polar coordinates of the relative velocity. I'm happy enough with the cartesian coordinates.

2. Example 4.14: Flying a Plane in a Wind.

   A pilot must fly a plane due north to reach their destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. (a) In what direction must the pilot head the plane to fly due north? (b) What is the speed of the plane relative to the ground?